ZERO-POINT KINETIC ENERGY OF RELATIVISTIC FERMION GASES

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The general idea of the geometrization of the relativistic phase-space suggested previously is discussed in some details and the method will be denoted as hyper-geometrization. Then, after a new definition of the relativistic phase-space volume, the well-known method of the calculation of the zero-point kinetic energy of perfect fermion gases will be generalized for the case when the gaseous systems are in gravitational fields with stationary Riemannian metrics.

§. 1. *Introduction*

In order to explain the internal structure of physical fields and elementary particles, respectively, *i. e.* to understand the symmetry properties recently found in high-energy physics, we have proposed [1] to consider the relativistic phase-space as a geometrical background of physical processes. The general idea of this proposal has been suggested by the analogy between the geometry of Yukawa's bilocal theory of fields and the relativistic phase-space formalism as well as by the arguments in favour of the belief that the phase-space formalism would be adequate to represent the dynamic symmetries of elementary particle physics in a natural way.

Nevertheless, an investigation of the geometrical structure of the relativistic phase-space has been suggested — independently of the arguments mentioned above — also by the increasing interest in the relativistic generalisations of kinetic and statistic theories of gases having the plasma physics [2—9] and modern astrophysical theories [9, 10] in mind. More recently [11] it was demonstrated that the proposed geometrization of the phase-space formalism [1] can very advantageously be used as a geometrical background at the foundation of the relativistic gas theory.

In the present investigation a new attempt will be suggested to clarify our way of thinking and after a short rewiew of previous results [1, 11] the definition of the relativistic phase-space volume, the zero point kinetic energy of perfect fermion gases will be calculated in Riemannian spaces to resolve a problem of the neutrino astrophysics [9, 10].

§. 2. *On the hyper-geometrization of the relativisiic phase-space*

The concept of the space-time represents the geometrisation of the space and time relations of the physical systems considered. While in the framework of the special theory of relativity only the kinematic aspects of the space and time relations are characterized and the space-time continuum as an underlying geometrical background of the physical events has been considered, in Einstein's theory of gravitation the space-time continuum has a significantly different meaning, namely, its geometrical structure is determined by the gravitational interactions, therefore, the space-time continuum represents a geometrization of the gravitational field, too.

Somewhat analogous geometrization of the kinematic and dynamic relations has been introduced in the framework of the phase-space formalism of the non-relativistic gas theory suggested earlier by the kinetic and statistic theories of Boltzmann. This means, however, that having a relativistic generalization of the phase-space formalism in mind, the concept of the relativistic phase-space has to reveal the geometrization — in other terms: a geometrical mapping — of the space-time *and* dynamic relations of the physical systems considered.

More concretely: the concept of space-time means the geometrization of the space and time relations based on the four-dimensional group of co-ordinate transformations and the concept of the non-relativistic phase-space means a geometrization of the dynamical relations based on the contact transformations of dynamics. The requirement seems be natural that in the case of the relativistic kinetic and statistic theories of gases the geometrical structure of the relativistic phase-space has to reveal the geometrization of the space-time *and* dynamical relations of the considered system based again on the four-dimensional group of co-ordinate transformations as it will be formulated below.

If a relativistic phase-space formalism has to be developed, one must introduce in every point of the four-dimensional space-time continuum a local momentum space which is, however, only three-dimensional due to the familiar normalization condition of the four-momenta; hence, the relativistic phase-space of the gaseous particles is $(4+3)$-dimensional. In fact, the „points", *i. e.*, the „radius vectors" of the momentum-space are potentially arbitrary but, actually, as the tangents of the world lines of the particles they are governed in all points of the space-time by the equations of motions, *i. e.*, by the dynamical space and time relations of the system considered. Under these circumstances the momentum-space means the internal degrees of freedom of system dynamics localized at every point of the space-time and determined unambiguously by the potentially arbitrary initial values of motion.

In terms of the phase-space formalism the trajectories of the gaseous particles reflect their dynamical history and depend on the initial values mentioned. As a matter of fact, the dynamical relations of the system at every point of the space-time continuum are determined by the tangents of the trajectories (world-lines), *i. e.*, by the actual momentum of the particle considered. Therefore, the dynamical relations of the system depending on the dynamical history of its constituents can be geometrized only if a geometry could be introduced in which the geometrical quantities at every point of the space considered are dependent on the directions, too. Such a geometrical framework means the geometry of the general line-element space where all the geometrical quantities in every point $x \equiv \{x^\mu\}$ of the space are dependent on the homogeneous direction coordinates $v \equiv \{v^\mu\}$ [12].

In fact, beside the space and time relations, the dynamical relations of the systems are geometrized in the framework of the line-element geometry by their dependence on the directions which means a second step in the geometrization, therefore this idea may be denoted as a kind of „hyper-geometrization" based on a geometry of an adequate four-dimensional anisotropic space.

§. 3. *Invariant volume-element of the relativistic phase-space*

The natural geometrical model of an anisotropic space is the line-element geometry [1, 11, 12] which is the geometry of an ensemble of line-elements $\{x^\mu, v^\mu\}$ rather than of points as in current geometries [1]. $\{x^\mu\}$ mean the position coordinates of the line-elements and by the contravariant vector $\{v^\mu\}$ with the transformation law

$$x^{\mu'} = x^{\mu'}(x^\mu); \quad v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu \quad \left(\Delta \stackrel{\text{def}}{=} \det \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| \neq 0\right) \tag{3.1}$$

the direction of the line-element $(x, v) \equiv \{x^\mu, v^\mu\}$ is determined. Since only a direction is defined by the vector $\{v^\mu\}$, the components v^μ are not independent of each other and only their proportions have meaning; *i. e.*, they, as homogeneous direction co-ordinates, may be regarded.

The geometric anisotropy means that all of the geometrical quantities, particularly, the components of the metrical fundamental tensor $g_{\mu\nu}$ — the most important geometrical quantity determining the geometrical structure of the line-element space — depend not only on the space point $\{x^\mu\}$ but also on the homogeneous direction coordinates $v = \{v^\mu\}$, *i. e.*, after all on the line-element: $g_{\mu\nu} = g_{\mu\nu}(x, v)$ being a homogeneous function of the variable v^μ of zero degree.

The one-parametric sequence of the line-elements

$$x^\mu = x^\mu(t), \; v^\mu = v^\mu(t) \qquad (t_1 \leqq t \leqq t_2) \tag{3.2}$$

is defined as a curve of the space $\mathscr{L}$ for the direction field $v^\mu(t)$. The length of the arch of our curve $x^\mu = x^\mu(t)$ for the regarded fields of directions $v^\mu = v^\mu(t)$ is defined by

$$s = \int_{t_1}^{t_2} \{g_{\mu\nu}(x, v) \dot{x}^\mu \dot{x}^\nu\}^{1/2} dt \quad \left(\dot{x}^\mu \stackrel{\text{def}}{=} \frac{dx^\mu}{dt}\right). \tag{3.3}$$

Now, let a scalar, so-called *fundamental function*

$$F(x, v) \stackrel{\text{def}}{=} \{g_{\mu\nu}(x, v) v^\mu v^\nu\}^{1/2} \tag{3.4}$$

be introduced, which is a homogeneous function of the variables v^μ of first order, then a vector of unit length can be defined in the direction of the line-element (x, v) by

$$l^\mu \stackrel{\text{def}}{=} v^\mu / F \tag{3.5}$$

which is naturally also a homogeneous function of zero degree of v^μ-s.

Due to the well known normalization condition of the four-momenta mentioned above[2]:

$$g_{\mu\nu} p^\mu p^\nu = m_0^2 \tag{3.6}$$

by the components p^μ of the four-momenta merely a direction is determined in the space-time continuum, therefore the identification

$$l^\mu \equiv p^\mu / m_0 \tag{3.7}$$

[1] The Greek indices have to be run over 0, 1, 2, 3 and the Latin ones over 1, 2, 3. Further, Einstein's summation convention will be used.

[2] The velocity of light *in vacuo* as the unit of velocity is introduced!

may be suggested and the (4+3)-dimensional set of "points" $\{x^\mu, p^\mu\}$ as the relativistic *phase-space* can be regarded. As usual, let the ensemble of the co-ordinates $\{x^\mu\}$ be called as *co-ordinate-space* or — in order to avoid any confusion with the space part of the space-time — *configuration-space* and that of the momentum components $\{p^\mu\}$ in every point $\{x^\mu\}$ of the configuration space as *momentum-space*. This means, however, that the framework of the line-element geometry is indeed a well fitted geometrical basis for the relativistic gas theories.

The components of the metrical foundamental tensor are in pseudo-Euclidian space-time usually constant (with the signature: $+ \; - \; - \; -$) and in Riemannian space they are depending on the co-ordinates $\{x^\mu\}$. Now, if a certain direction is distinguished in the momentum-space — *e. g.*, the considered gaseous system streams with constant velocity — the momentum-space and the phase-space, respectively, loses its isotropy and the components of the metrical fundamental tensor depend on the homogeneous direction co-ordinates $\{v^\mu\}$, too; *i. e.*, $g_{\mu\nu} = g_{\mu\nu}(x^\mu, v^\mu) = g_{\mu\nu}(x^\mu, p^\mu)$. This general case will, however, not be treated below [1, 11] and actually only the Riemannian metric $g_{\mu\nu} = g_{\mu\nu}(x^\mu)$ will be used.

In order to define the local momentum-space in a covariant way under the group $\mathscr{G}_x$ of the general co-ordinate transformations (3.1), first of all let the *inhomogeneous direction co-ordinates*

$$\xi_i \overset{\text{def}}{=} \underset{i}{\lambda}^\mu p_\mu / m_0 \tag{3.8}$$

be introduced referring them at any point $\{x^\mu\}$ of the configuration-space to an orthonormal triad with space-like axes $\underset{i}{\lambda}^\mu$ $(i=1, 2, 3)$, then the orientations of the triads at different points of the configuration-space has to be co-ordinated.

The orthonormality of the space-like triad-axes means, of course,

$$g_{\mu\nu} \underset{i}{\lambda}^\mu \underset{k}{\lambda}^\nu = -\delta_{ik} \qquad (i, k = 1, 2, 3). \tag{3.9}$$

One can immediately see that if the triad-axes $\underset{i}{\lambda}^\mu$ are in the rest system $\mathscr{K}^0$ of the gaseous particles orthogonal to the four-momentum $\{p^\mu_{(0)}\} \equiv \{p^0_{(0)}, 0, 0, 0\}$, *i. e.*

$$\underset{i}{\lambda}^\mu_{(0)} p_{(0)\mu} = 0 \qquad (i = 1, 2, 3) \tag{3.10}$$

(where $\underset{i}{\lambda}^\mu_{(0)}$-s denote the components of the triad axes in the frame $\mathscr{K}^0$) and the sign of p^0 — corresponding to the sign of the energy of the particles — is fixed, the inhomogeneous direction co-ordinates, being the cosine of the angles between the radius vector p^μ and the triad-axes $\underset{i}{\lambda}^\mu$, *i. e.*

$$\vartheta_i \overset{\text{def}}{=} \arccos \{\underset{i}{\lambda}^\mu p_\mu / m_0\} \tag{3.11}$$

determine uniquely the directions characterized by the momentum vectors $\{p^\mu\}$ at the points $\{x^\mu\}$ of the configuration-space.

Let the three by pairs orthogonal unit vectors $\underset{i}{\lambda}^\mu$ be called in the following as *λ-triad*. Of course, the λ-triad as a *new local frame of reference* and the inhomogeneous direction co-ordinates as the *independent components of the radius vectors of the momentum-space* may be used.

Owing to the definition (3.8) of the inhomogeneous direction co-ordinates they are invariants of the transformations (3.1) of the group $\mathscr{G}_x$. Nevertheless, the ξ_i-s are not only depending on the momentum components p^μ but also on the *a priori* orientation of the λ-triad introduced. As a matter of fact, by changing the directions of the triad-axes intrinsic transformations of the local momentum-space can by introduced which cannot be induced by any change of the co-ordinates in the configuration-space. By these transformations of the momentum-space — denoted further by $\mathscr{G}_\xi$ — internal dynamical degrees of freedom of the gas system can be characterized. Therefore, let the „co-ordinates" $\{\xi_i\}$ of the local momentum-space be called as *internal co-ordinates* and the "co-ordinates" $\{x^\mu\}$ of the configuration-space as *external co-ordinates* of the system considered.

In order to determine the handedness of the λ-triad unambigously, keeping the relations (3.9) and (3.10) in mind, it seems to be deemed proper to suppose that in the rest frame of reference $\mathscr{K}^0$ in the configuration-space the axes of the right-handed λ-triad have consecutively the same directions as those of the spatial axes of the original tetrad distinguishing $\mathscr{K}^0$; *i. e.*, they are explicitly given by

$$\{\underset{1}{\lambda}{}^\mu_{(0)}\} \equiv \{0, 1, 0, 0\}, \quad \{\underset{2}{\lambda}{}^\mu_{(0)}\} \equiv \{0, 0, 1, 0\}, \quad \{\underset{3}{\lambda}{}^\mu_{(0)}\} \equiv \{0, 0, 0, 1\}. \tag{3.12}$$

Let the right-handed λ-triad be called in the following as λ^+*-triad*, and the left handed one as λ^-*-triad*. It is obvious that by settling the handedness of the λ-triads in this way the vectors $\underset{i}{\lambda}{}^\mu$ are unambigously determined in any frame of reference.

This is the reason that, for sake of simplicity, our argumentation will be mostly developed in the system $\mathscr{K}^0$; namely, the results obtained can be transformed into all frames of reference without any difficulty. However, we have to bear, in all circumtances, in mind that a special choice of the orientation of the λ-triad means simultaneously a special choice of a distinguished frame of reference in the configuration-space. In fact, the rest frame of reference $\mathscr{K}^0$ meets a natural distinction of the frames; this is the reason for favouring it in the following.

As a matter of fact, the group $\mathscr{G}_\xi$, *i. e.*, the group of internal transformations of the momentum-space can be generated by the rotations of the λ-triad around its origin and by the reflexions on different symmetry elements of the triad. Consider, first of all, the rotations of the λ-triad which can be characterized by the Eulerian angles $\{\varphi, \psi, \vartheta\}$:

$$\underset{i}{\lambda}{}'^\mu = M_i^k \underset{k}{\lambda}{}^\mu, \tag{3.13}$$

where the well-known matrix-elements $M_i^k = M_i^k(\varphi, \psi, \vartheta)$ fulfil the orthogonality relations

$$M_i^k M_j^k = \delta_{ij} \quad \text{and} \quad M_i^r M_i^s = \delta^{rs}. \tag{3.14}$$

Due to the definition (3.8) of the internal (inhomogeneous direction) co-ordinates their transformation law can be obtained as follows:

$$\xi_{i'} = \underset{i}{\lambda}{}'^\mu p_\mu / m_0 = M_i^k \underset{k}{\lambda}{}^\mu p_\mu / m_0 = M_i^k \xi_k. \tag{3.15}$$

More generally we have:

$$\xi_i' = D_i^k \xi_k. \tag{3.16}$$

As the orthogonality relations (3. 10) have to remain valid also the new triad axes

$\lambda'^{\mu}_{(0)}$ are orthogonal to $p^{\mu}_{(0)}$, therefore the group $\mathscr{G}_{\xi}$ is isomorphic to the three-dimensional rotary-reflexion group.

In order to settle the orientation of the λ-triad at any or at different points of the space-time continuum it seems to be simple to use the framework of the orthonormal tetrad formalism of the Riemannian spaces discussed in details by SYNGE [13] especially favourable for our purposes.

As an orthonormal tetrad, four by pairs orthogonal unit vectors $\Lambda^{\mu}_{(\alpha)}$ are denoted where the indices in parantheses like (α) mean a label distinguishing the particular axes. The covariant components of the same tetrad are

$$\Lambda_{(\alpha)\mu}=g_{\mu\nu}\Lambda^{\nu}_{(\alpha)}. \tag{3.17}$$

Three of the axes are, of course, space-like and one is time-like. We shall always so label the axes that $\Lambda^{\mu}_{(0)}$ is time-like.

The conditions of orthonormality may be written in the form:

$$\Lambda^{\mu}_{(\alpha)}\Lambda_{(\beta)\mu}=\eta_{(\alpha\beta)}, \tag{3.18}$$

where

$$\eta_{(00)}=-\eta_{(11)}=-\eta_{(22)}=-\eta_{(33)}=\eta,\quad \eta_{(\alpha\beta)}=0\qquad (\alpha\neq\beta) \tag{3.19}$$

$$\eta_{(\alpha\beta)}=\eta^{(\alpha\beta)}$$

is a diaxonal matrix; it satisfies the relation

$$\eta^{(\alpha\beta)}\eta_{(\beta\gamma)}=\delta^{\alpha}_{\gamma} \tag{3.20}$$

being, in language of matrix algebra, a square root of unity.

One has to emphasize that the labels on the vectors have no tensorial meaning; nevertheless, by means of the η-matrix the framework of the tensor calculus can be introduced. Let the raising and lowering of the labels be defined by

$$\Lambda^{(\alpha\mu)}=\eta^{(\alpha\beta)}\Lambda^{\mu}_{(\beta)}\quad\text{and}\quad \Lambda^{(\alpha)}_{\mu}=\eta^{(\alpha\beta)}\Lambda_{(\beta)\mu}, \tag{3.21}$$

then owing to eqs. (3.20) we have

$$\Lambda^{\mu}_{(\alpha)}=\eta_{(\alpha\beta)}\Lambda^{(\beta)\mu}\quad\text{and}\quad \Lambda_{\alpha(\mu)}=\eta_{(\alpha\beta)}\Lambda^{(\beta)}_{\mu}, \tag{3.22}$$

respectively. Finally, the relations

$$\Lambda^{\mu}_{(\alpha)}\Lambda^{(\beta)}_{\mu}=\delta^{\beta}_{\alpha}\quad\text{and}\quad \Lambda^{\mu}_{(\alpha)}\Lambda^{(\alpha)}_{\nu}=\delta^{\mu}_{\nu} \tag{3.23}$$

can be obtained. The two tetrads $\Lambda^{(\alpha)}_{\mu}$ and $\Lambda^{(\alpha)\mu}$ are closely connected: their space-like axes are the same and their time-like ones are opposed to one another, *i. e.*, they are different in their handedness.

Let us give at a space-time point $\{x^{\mu}\}$ two orthonormal tetrads, $\Lambda^{\mu}_{(\alpha)}$ and $\Lambda'^{\mu}_{(\alpha)}$, they can be connected by a Lorentz transformation with the so-called *Lorentz matrix*

$$L^{(\alpha)}_{(\alpha)'}\overset{\text{def}}{=}\Lambda^{(\alpha)}_{\mu}\Lambda'^{\mu}_{(\alpha)} \tag{3.24}$$

being the unit matrix if the two tetrads coincide. Owing to eqs. (3.23) and (3.24) at every space-time point the equivalent Lorentz transformations:

$$\Lambda'^{\mu}_{(\alpha)}=L^{(\alpha)}_{(\alpha)'}\Lambda^{\mu}_{(\alpha)}\quad\text{and}\quad \Lambda^{(\alpha)}_{\mu}=L^{(\alpha)}_{(\beta)'}\Lambda'^{(\beta)}_{\mu} \tag{3.25}$$

can be introduced, being independent of any changes of the space-time co-ordinates. These Lorentz transformations may be interpreted as the "internal" changes of the orientation of the tetrads.

Now, let us suppose that in the local rest frame of reference $\mathscr{K}^0$ of the gaseous particles, distinguished by an original tetrad $\Lambda^{*\mu}_{(\alpha)}$, we have

$$p^{\mu}_{(0)} \overset{\text{def}}{=} \Lambda^{*\mu}_{(0)}, \quad \underset{i}{\lambda}^{\mu} \overset{\text{def}}{=} \Lambda^{*\mu}_{(i)} \qquad (i = 1, 2, 3), \tag{3.26}$$

then one can immediately see that the λ^+-triad as the space-like part of the Lorentz-covariant $\Lambda^{*\mu}_{(\alpha)}$ tetrad and the λ^--triad as the space-like part of the Lorentz-contravariant $-\Lambda^{*(\alpha)\mu}$ tetrad has to be defined.

As a matter of fact, any changes of the orientation of the λ-triad generating the group of internal transformations $\mathscr{G}_{\xi}$, is the spatial subgroup of the infernal Lorentz transformations (3.24).

Associated with each point of a curve $x^{\mu}=x^{\mu}\sigma$ in space-time an orthonormal tetrad can be introduced with particularly considerable features formed by the *unit tangent*

$$t^{\mu}_{(0)} \overset{\text{def}}{=} \frac{dx^{\mu}}{d\sigma}, \tag{3.27}$$

as well as by the *first, second* and *third normals to the curve* denoted by $\eta^{\mu}_{(1)}$, $\eta^{\mu}_{(2)}$ and $\eta^{\mu}_{(3)}$, respectively. These by pairs orthogonal unit vectors are determined by means of the well-known Frenet-Serret formulae:

$$\begin{cases} \dfrac{D}{d\sigma} t^{\mu}_{(0)} = \varrho_1 n^{\mu}_{(1)}, & \dfrac{D}{d\sigma} n^{\mu}_{(2)} = \varrho_3 n^{\mu}_{(3)} - \varrho_2 n^{\mu}_{(1)}, \\ \dfrac{D}{d\sigma} n^{\mu}_{(1)} = \varrho_2 n^{\mu}_{(2)} + \varrho_1 t^{\mu}_{(0)}, & \dfrac{D}{d\sigma} n^{\mu}_{(3)} = -\varrho_3 n^{\mu}_{(2)}, \end{cases} \tag{3.28}$$

where the scalars, ϱ_1, ϱ_2 and ϱ_3 are the first, second and third curvatures of the curve considered. In the case of time-like curves, *i. e.*, in the case of curves with time-like unit tangents, we have:

$$g_{\mu\nu} t^{\mu}_{(0)} t^{\nu}_{(0)} = 1, \quad g_{\mu\nu} n^{\mu}_{(i)} n^{\nu}_{(i)} = -1 \qquad (i = 1, 2, 3). \tag{3.29}$$

This so-called *normal tetrad* $\{t^{\mu}_{(0)}, n^{\mu}_{(i)}\}$ *to the curves* will be used below when the orientation of the tetrads at different distinct space-time points will be compared.

Considering at any two distinct points of the Riemannian space-time continuum a tetrad and λ-triad, respectively, their orientation has to be compared by means of the framework of the general parallel transport along the world lines of the particles.

The world lines of the particles are time-like curves with equations $x^{\mu}=x^{\mu}(\sigma)$. It is well-known that a vector is said to undergo *parallel transport* along a curve if its absolute derivative vanishes.

$$\frac{DV^{\mu}}{d\sigma} \overset{\text{def}}{=} \frac{dV^{\mu}}{d\sigma} + \left\{ {\mu \atop \varkappa \cdot \lambda} \right\} V^{\varkappa} \frac{dx^{\lambda}}{d\sigma} = 0. \tag{3.30}$$

In the following it seems to be more favoured to use a particular kind of parallel transports of a vector V^{μ} — called usually Fermi—Walker transport ([14, 15] and [13] as well) — along the world lines of the particles defined by the equation

$$\frac{DV^{\mu}}{d\sigma} = \frac{\varrho_1}{m_0} V_{\varkappa}(p^{\mu} n^{\varkappa}_{(1)} - p^{\varkappa} n^{\mu}_{(1)}), \tag{3.31}$$

where attention was paid to the fact that in the case of particles world lines $x^{\mu}=x^{\mu}(\sigma)$ the unit tangent is just the four-momentum of the particles normalized to unity:

$$t^{\mu}_{(0)} = \frac{dx^{\mu}}{d\sigma} = p^{\mu}/m_0. \tag{3.32}$$

The important features of the Fermi—Walker transport are that

(*i*) the unit tangent $t^{\mu}_{(0)}$ itself automatically undergoes Fermi—Walker transport, as it can be checked on the basis of eqs. (3.31) and (3.28) immediately;

(*ii*) it resembles parallel transport in the conservation of magnitude and scalar product;

(*iii*) if the Fermi—Walker transport is applied to the normals $m_{(i)}$ which are orthogonal to the tangent $t^{\mu}_{(0)}$ at some point of the curve considered, they remain, of course, orthogonal to $t^{\mu}_{(0)}$ and to each other. This means, however, that the normal tetrad $\{t^{\mu}_{(0)}, n^{\mu}_{(i)}\}$ under Fermi—Walker transport remains normal tetrad along any curves.

As a matter of fact, the comparison of the orientations of two orthonormal tetrads at two distinct points $\{x^{\mu}\}$ and $\{y\}^{\mu}$ of the Riemannian space-time can be mastered in the following way:

Let the two considered points be connected by a world-line with a unit tangent the direction of which, *e. g.*, at the point $\{x^{\mu}\}$ coincides with that of the $t^{\mu}_{(0)}$ axis of the tetrad considered. Then, let the orthonormal tetrad with its origin, *e. g.*, at $\{x^{\mu}\}$ be undergone Fermi—Walker transport as long as its origin coincides with $\{y^{\mu}\}$. In this way a virtual tetrad is unambiguously oriented, which, as a basis to determine the orientation of the second tetrad — with its origin originally at $\{y^{\mu}\}$ — can be used by means of the method for the comparison of tetrads at the same space-time point in terms of the internal Lorentz transformations.

The comparison of the λ-triads of different space-time points, having their definition (3.26) in mind, based on that of the tetrads is straightforward and we have not to enter upon it. However, then the inhomogeneous direction co-ordinates $\{\xi_i\}$ of the local momentum-spaces, originally defined at different space-time points, can be „synchronized" and the relativistic phase-space in Riemannian space-time can be defined as a direct product of the configuration- and the momentum-spaces, respectively.

Finally, we have to mention that the definition of the inhomogeneous direction co-ordinates — the framework of which, having the invariant characterization of the internal degrees of freedom of physical fields in mind, was suggested several years ago [16] — seems to be very close to that of the spatial set of Fermi co-ordinates [14] the advantages of which from other points of view were emphasized by SYNGE [13].

In order to define the invariant volume-element of the relativistic phase-space, we have (*i*) to keep in mind the familiar definition of the corresponding scalar density of the Riemannian space-time, (*ii*) to propose a reasonable scalar density in the momentum-space, then we have the possibility to construct an adequate invariant under the transformations of a certain group to be defined below. We would have several possibilities to carry out this program. However, it seems to be reasonable to accept such a definition of the relativistic phase-space volume-element which at a certain instant, *i. e.*, on a space-like surface of the space-time continuum is reduced into the familiar phase-space volume-element of the non-relativistic theory.

As a matter of fact, we have to suggest the following definition of the hyper-surfaces in the framework of general line-element geometry:

As hyper-surface of the geometrized relativistic phase-space the ensemble of the line-elements $\{x^{\mu}=x^{\mu}(u^i);\, p^{\mu}\}$ *or* $\{x^{\mu}=x^{\mu}(u^i);\, \xi_i\}$ $(i=1,2,3)$ *will be denoted, where* $\{u^i\}$ *and* $u^i=\text{const.}$ *mean respectively the parameters and the parametric lines of the three-dimensional hyper-surface of the co-ordinates of the line-elements.*

Let the set of quantities

$$\varepsilon_{\alpha\beta\gamma\delta} \stackrel{\text{def}}{=} \begin{cases} +1, & \textit{if } \{\alpha, \beta, \gamma, \delta\} \textit{ means an even permutation of the numbers } \{0, 1, 2, 3\}, \\ -1, & \textit{if } \{\alpha, \beta, \gamma, \delta\} \textit{ means an odd permutation of the numbers } \{0, 1, 2, 3\}, \\ 0, & \textit{if at least two of the indices } \{\alpha, \beta, \gamma, \delta\} \textit{ agree} \end{cases} \tag{3.33}$$

be introduced, which are *per definitionem* anti-symmetric in all their indices, and let the pseudo-tensor

$$\eta_{\alpha\beta\gamma\delta} \stackrel{\text{def}}{=} \sqrt{-g}\, \varepsilon_{\alpha\beta\gamma\delta} \quad (g \stackrel{\text{def}}{=} \det |g_{\mu\nu}|) \tag{3.34}$$

with the law of transformation

$$\eta_{\alpha'\beta'\gamma'\delta'} = \operatorname{sgn}\{\Delta\} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \frac{\partial x^\delta}{\partial x^{\delta'}} \eta_{\alpha\beta\gamma\delta} \tag{3.35}$$

be defined, then the normal vector to the hyper-surface of the configuration-space in the form

$$\nu_\alpha \stackrel{\text{def}}{=} \eta_{\alpha\beta\gamma\delta} \frac{\partial x^\beta}{\partial u^1} \frac{\partial x^\gamma}{\partial u^2} \frac{\partial x^\delta}{\partial u^3} \tag{3.36}$$

can be obtained, where $\partial x^\beta / \partial u^i$ mean the tangents to the parametric lines $u^i = \text{const.}$ of the surface considered.

The length of the normal vector ν_α one calculates in a straightforward way:

$$\nu_\alpha \nu^\alpha = g^* \tag{3.37}$$

with

$$g^*_{ik} \stackrel{\text{def}}{=} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^k} \quad \text{and} \quad g^* \stackrel{\text{def}}{=} \det\, |g^*_{ik}|. \tag{3.38}$$

This means, however, that the unit normal vector to the hyper-surface $\{x^\mu = x^\mu(u^1, u^2, u^3); p^\mu\}$ is given by

$$n_\alpha = \nu_\alpha / \sqrt{|g^*|}. \tag{3.39}$$

Let us suppose in the following that the vectors ν_α and n_α, respectively, are time-like vectors, *i. e.*, $g^* > 0$; then the set of tangents of the hyper-surface and the hyper-surface itself will be called space-like.

Keeping this familiar definitions in mind, the *oriented hyper-surface elements* can be introduced also in the case *of the relativistic* configuration-space by means of the definition

$$df_\varrho \stackrel{\text{def}}{=} \nu_\varrho\, du^1 du^2 du^3 = n_\varrho\, df \tag{3.40}$$

with

$$df \stackrel{\text{def}}{=} \sqrt{|g^*|}\, du^1 du^2 du^3 \tag{3.41}$$

being the invariant measure of the hyper-surface element.

Considering the curve $x^\varrho = x^\varrho(s)$ of the configuration-space (s being again the parameter of the length of arch of the line-element geometry), let us suppose that *the unit tangent of the curve coincides in the crossing point of the curve and the hyper-surface with the unit normal vector of the hyper-surface, i. e.*,

$$\frac{dx^\varrho}{ds} = n^\varrho \quad \text{or} \quad dx^\varrho = n^\varrho\, ds, \tag{3.42}$$

then the analytical definition of *the invariant volume-element of the configuration-space can in general be given by*

$$dV \stackrel{\text{def}}{=} \frac{\sqrt{-g}}{\sqrt{|g^*|}}\, dx^\varrho\, df_\varrho = \frac{\sqrt{-g}}{\sqrt{|g^*|}}\, n^\varrho n_\varrho\, df\, ds = \frac{\sqrt{-g}}{\sqrt{|g^*|}}\, df\, ds. \tag{3.43}$$

In the particular important special case of the parametrization

$$x^0 \equiv s,\ x^i \equiv u^i \qquad (i = 1, 2, 3) \tag{3.44}$$

the formally well-known formula:

$$dV = \sqrt{-g}\, dx^0\, dx^1\, dx^2\, dx^3 \stackrel{\text{def}}{=} \sqrt{-g}\, d^4 x \tag{3.45}$$

can be obtained.

Due to the definition (3.8) of the inhomogeneous direction co-ordinates $\{\xi_i\}$, they are invariants of the group of the transformations of the co-ordinates. However, if instead of the line-element $\{x^\mu, p^\mu\}$, the line-element $\{x^\mu, p^\mu + Dp^\mu\}$ is considered, the inhomogeneous direction co-ordinates are, of course, changed and in the general (anisotropic) cases it is rather troublesome to determine their infinitesimal changes. But, in the special case when the metrical fundamental tensor does not depend on the direction co-ordinates, one can immediately see that

$$d\xi_i = m_0^{-1} \underset{i}{\lambda}{}^\mu \{(p_\mu + Dp_\mu) - p_\mu\} = Dp_\mu \underset{i}{\lambda}{}^\mu / m_0, \tag{3.46}$$

where Dp_μ denotes the covariant differential of p_μ.

Owing to the obvious invariance of $d\xi_i$ against any co-ordinate transformation, the invariant volume-element of the local momentum-space at arbitrary but fixed point of the configuration-space can, of course, be defined as follows:

$$dP \stackrel{\text{def}}{=} m_0^3\, d\xi_1\, d\xi_2\, d\xi_3 \stackrel{\text{def}}{=} m_0^3\, d^3\xi, \tag{3.47}$$

where the factor m_0^3 has to be introduced in order to save the correct physical dimensions of the volume-element of the momentum-space.

Although dP is an invariant of the group $\mathscr{G}_x$, it will generally change if the internal transformations

$$\xi_{i'} = \xi_{i'}(\xi_i) \tag{3.48}$$

of the group $\mathscr{G}_\xi$ are considered. Bearing in mind that the transformations $\mathscr{G}_\xi$ are homogeneous linear orthogonal transformations — *i. e.*, they are isomorphic to the three-dimensional subgroup of the Lorentz transformations (3.24) — of the type

$$\xi_{i'} = D_{i}^{k} \xi_{k'} \tag{3.49}$$

we have

$$d^3\xi' \equiv d_1'\xi\, d_2'\xi\, d_3'\xi = \frac{\partial(\xi_1', \xi_2', \xi_3')}{\partial(\xi_1, \xi_2, \xi_3)}\, d\xi_1\, d\xi_2\, d\xi_3 = \operatorname{sgn}\{D\}\, d^3\xi \tag{3.50}$$

and, as a matter of fact, dP will be under the group $\mathscr{G}_\xi$ of the internal transformations a pseudo-scalar. Indeed, the invariant volume-element of the local momentum-space depends on the orientation of the basic λ-triad.

The underlying general group $\mathscr{G}$ in the background of the concept of the relativistic phase-space — corresponding to the relativistic generalization of the group of the contact transformations of classical dynamics — is, of course, the direct product of the groups of the external and internal transformations, *i. e.*,

$$\mathscr{G}=\mathscr{G}_x \times \mathscr{G}_\xi . \tag{3.51}$$

This means, however, that as the relativistic phase-space volume-element in terms of the parametrization (3.44) the expression

$$d\Omega \stackrel{\mathrm{def}}{=} m_0^3 \sqrt{-g}\, d^4x\, d^3\xi \tag{3.52}$$

can be introduced being a pseudo-scalar of the group $\mathscr{G}$. The scalar factor m_0^3 is again considered to keep the correct physical dimensions of the phase-space volume-element.

In fact, the relativistic phase-space volume-element may be oriented; let it be denoted as a positive one if the underlying λ-triad is right-handed.

At a given instant of time, *i. e.*, on the hyper-plain $x^0=\mathrm{const.}$ of the configuration-space, the phase-space volume-element is reduced into the form

$$d\Omega_0 \stackrel{\mathrm{def}}{=} m_0^3 \sqrt{-g}\, dx^1 dx^2 dx^3 d\xi_1 d\xi_2 d\xi_3 = m_0^3 \sqrt{-g}\, d^3x\, d^3\xi \tag{3.53}$$

being the direct generalization of the well-known expression of the non-relativistic gas theory. For sake of simple speaking $d\Omega_0$ will be called in the following as the *momentary expression of the phase-space volume-element.*

Finally, owing to the general definition (3.8) of the inhomogeneous direction co-ordinates $\{\xi_i\}$, let the explicit forms of the relativistic phase-space volume-element be obtained in two important cases, in special frames of reference defined in different underlying metrical space-times:

(*i*) Considering a pseudo-Euclidian space-time continuum with the metrical fundamental tensor

$$\gamma_{00} = -\gamma_{11} = -\gamma_{22} = -\gamma_{33} = 1; \quad \gamma_{\mu\nu} = 0 \quad (\mu \neq \nu), \tag{3.54}$$

let us first suppose that the axes of the λ^+-triad due to its orientation in the rest frame of reference $\mathscr{K}^0$ are given by (3.12), then based on the definitions (3.46) of the scalar differentials $d\xi_i$ of the inhomogeneous direction co-ordinates one obtains

$$d\xi_i = -dp^i/m_0 = dp_i/m_0. \tag{3.55}$$

This means, however, that the relativistic phase-space volume-element is given in this case by

$$d\Omega = dx^0\, dx^1\, dx^2\, dx^3\, dp_1\, dp_2\, dp_3 \stackrel{\mathrm{def}}{=} d^4x\, d^3p_{\mathrm{cov}} \tag{3.56}$$

and its momentary expression, *i. e.*, its expression on the $x^0=\mathrm{const.}$ hyper-plain can be put in the form

$$d\Omega_0 = d^3x\, d^3p_{\mathrm{cov}}; \tag{3.57}$$

the very familiar expression of the non-relativistic phase-space volume-element.

(*ii*) In our case, investigated below, the background space-time continuum is Riemannian. In order to obtain the phase-space volume-element and its momentary expression, respectively, one has to use explicitly the framework of the theory of external forms [6, 12]. Namely, in terms of the method of the external forms, the phase-space volume-element can be written as follows:

$$d\Omega = m_0^3 \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \tag{3.58}$$

with the temporarily more favourable abbreviation of the commutators, *e. g.*, A and B:

$$A \wedge B \stackrel{\mathrm{def}}{=} AB - BA. \tag{3.59}$$

Having the above introduced frame of reference and orientation of the λ-triad in mind we have

$$d\xi_i = Dp_i = m_0^{-1}(dp_i - \{_i{}^\mu{}_\lambda\} p_\mu dx^\lambda), \tag{3.60}$$

where $\{_i{}^\mu{}_\nu\}$ denotes the components of the Christoffel-symbols. Nevertheless, due to identity

$$dx^\mu \wedge dx^\nu \equiv 0, \tag{3.61}$$

finally the expressions

$$d\Omega = \sqrt{-g}\, d^4x\, d^3p_{\text{cov}} \tag{3.62}$$

and

$$d\Omega_0 = \sqrt{-g}\, d^3x\, d^3p_{\text{cov}}, \tag{3.63}$$

respectively, can be obtained.

§. 4. *The zero-point kinetic energy of perfect fermion gases*

In addition to the natural theoretical interest in the generalization of important physical concepts, the investigations to deal with the definitions of the energy and momentum of particles on the Fermi level in the case of relativistic fermion gases in Riemannian space-time continuums are also suggested by problems more recently raised in the neutrino-astronomy [9, 10].

Considering a *completely degenerate fermion gas, i. e.,* — for the sake of simplicity — a perfect system of fermions at zero point of the absolute temperature, the fermions are distributed over the different quantum states in such a way that the total energy E of the gas has its smallest possible value. Due to the fundamental feature of fermion gases, — *i. e.,* to Pauli's exclusion principle,— no more than one fermion may be in any one state, the fermions fill all states with energies between the smallest (equal to zero) and some largest value which is determined by the number N of the fermions. Let us suppose that the fermion gas is in a volume V_0 of the configuration space, then the smallest possible value of its total energy — the so-called *zero point kinetic energy* E_0 — can simply be calculated based on the framework of the non-relativistic phase-space method having in the background the Euclidian geometry in mind.

Let us suppose that the intrinsic dynamical properties of the completely degenerate fermion gas considered are isotropic, then the number of quantum states of translational motion of a particle with a momentum whose absolute value lies between $\mathbf{p} \equiv p$ and $p+dp$ is:

$$dn = 4\pi p^2\, dp\, dV_0/(2\pi\hbar)^3, \tag{4.1}$$

where the familiar convention has been taken into account, that the volume of the phase-space corresponding to each degree of freedom is just equal to the Planck's constant $h \equiv 2\pi\hbar$.

Let the highest value of the energy be called as *Fermi energy* ε_F, the corresponding momentum as *Fermi momentum* p_F as well as the energy state itself as *Fermi-level,* then the number of filling all states in the momentum interval $(0, p_F)$ is given by

$$N = \gamma V_0 (2\pi\hbar)^{-3} \int_0^{p_F} dp^2\, dp = \gamma V_0 p_F^3/6\pi^2\hbar^3, \tag{4.2}$$

where γ denotes the number of the internal degrees of freedom in each quantum state (having usually only the spin degeneracy in mind). The only unknown quantity in relation (4.2) is the Fermi momentum; therefore, we have

$$p_F = \hbar(6\pi^2 N/\gamma V_0)^{1/3} \equiv \hbar(6\pi^2 \varrho/\gamma)^{1/3} \tag{4.3}$$

based on which in the familiar way for the Fermi energy

$$\varepsilon_F = \frac{p_F^2}{2m_0} = \frac{\hbar^2}{2m_0}\left(\frac{6\pi^2 N}{\gamma V_0}\right)^{2/3} = \frac{\hbar^2}{2m_0}\left(\frac{6\pi^2}{\gamma}\varrho\right)^{2/3} \tag{4,4}$$

and for the zero point kinetic energy

$$E_0 = \frac{3\hbar^2}{10m_0} N \left(\frac{6\pi^2}{\gamma} \bar{\varrho} \right)^{2/3}, \tag{4.5}$$

respectively, can be obtained, where

$$\bar{\varrho} = N/V_0 \tag{4,6}$$

denotes the *mean value of the density of fermions* in the configuration-space.

Keeping in mind the general definitions (3.62) and (3.63) of the relativistic phase-space volume-element and its momentary value, respectively, all of the classical concepts and results summarized above may be directly generalized for a relativistic and dynamically anisotropic system of fermions in space-time continuum with general structure, too, characterized by the metrical fundamental tensor $g_{\mu\nu}(x, p)$.

However, for the sake of simplicity let the relativistic neutrino gas be considered in Riemannian spaces with spherical symmetry. First the somewhat more general case of fermions with non-vanishing rest mass, $m_0 \neq 0$, will be discussed, then the results will be specialized for the neutrino gas by the limiting process $m_0 \to 0$.

Owing to the classical results summarized above the Fermi momentum and energy of the gaseous particles depend on the mean density N of fermions in the three-dimensional configuration-space. This mean density can naturally be defined either in a closed universe or in open universes having in mind the framework of infinite systems of fermions, *i. e.*, the limits $N \to \infty$ and $V_0 \to \infty$ with the restriction $\bar{\varrho} = \text{const.}$ Both cases will be discussed in Riemannian space-time continuums with particularly interesting metrical structures.

Due to the spherical structure of the space-time continuum and to the dynamical isotropy of systems of the considered particles on the $x^0 = \text{const.}$ hyper-plain the spherical polar co-ordinates $\{r, \vartheta, \varphi\}$ and $\{p, \theta, \varphi\}$ will, of course, be introduced in the configuration- and local momentum-spaces, respectively. This means that the momentary relativistic phase-space volume of the degenerate relativistic fermion gas considered has to be defined by

$$\Omega_0 = \int_{\alpha}^{R} dr \int_0^{\pi} d\varphi \int_0^{2\pi} d\varphi \int_0^{p_F} dp \int_0^{\pi} d\theta \int_0^{2\pi} d\Phi \sqrt{-g}\, p^2 \sin\theta, \tag{4.7}$$

where the upper limit R of the configuration-space integral means either the radius of the universe or in the case of open universe the spherical symmetric space-time domain taken into account before the limiting process $R = (3V_0/4\pi)^{1/3} \to \infty$ and the lower one is determined by its metrical properties.

Keeping in mind the volume-element dP of the local momentum-space defined by eq. (3.47), the absolute value of the three-momentum has to be calculated from the normalization condition of the covariant four-momentum components:

$$g^{\mu\nu}(x) p_\mu p_\nu = m_0^2, \tag{4.8}$$

since the momentum components with covariant transformation character have to be considered in the actual version (3.57) of dP.

In the particular cases considered in the following the components of the metrical fundamental tensor may generally be given as follows:

$$g_{00} = -h_0(r),\ g_{11}=h_1(r),\ g_{22}=h_2(r),\ g_{33}=h_3(r)\sin^2\vartheta;\ g_{\mu\nu}=0 \qquad (\mu\neq\nu). \tag{4.9}$$

The reason that here the signature $(-+++)$ was introduced instead of the usual one, is that the hyper-surface $x^0=\text{const.}$ is space-like. This means, however, that

$$g = -h_0h_1h_2h_3\sin^2\vartheta \quad \text{and} \quad \sqrt{-g} = \{h_0h_1h_2h_3\}^{1/2}\sin\vartheta, \tag{4.10}$$

respectively, furthermore

$$g^{00} = -h^{-1}(r),\quad g^{11} = h_1^{-1}(r),\quad g^{22} = h_2^{-1}(r),\quad g^{33} = h_3^{-1}(r)\sin^{-2}\vartheta; \tag{4.11}$$
$$g^{\mu\nu}=0 \qquad (\mu\neq\nu).$$

Therefore, the normalization condition can be put into the form:

$$-h_0^{-1}p_0^2+\{h_1^{-1}p_1^2+h_2^{-1}p_2^2+h_3^{-1}\sin^{-2}\vartheta p_3^2\} = -m_0^2 \tag{4.12}$$

and the definition of p may finally be given by

$$p \stackrel{\text{def}}{=} \{h_1^{-1}p_1^2+h_2^{-1}p_2^2+h_3^{-1}\sin^{-2}\vartheta p_3^2\}^{1/2} = \{\varepsilon^2h_0^{-1}(r)-m_0^2\}^{1/2}, \tag{4.13}$$

where it was taken into accont that in our system of units

$$p_0\equiv\varepsilon \tag{4.14}$$

denoting by ε the energy of the particles.

In order to introduce the inhomogeneous direction co-ordinates $\{\xi_i\}$ we have to normalize the basic vectors of the λ^+-triad. In terms of the parametrization (3.44) starting from the local rest frame of reference $\mathcal{K}^0$, *i. e.*, bearing the components of the triad axes (3.12) in mind, it can be obtained on the one hand

$$\begin{cases}\underset{1}{\lambda}^1=h_1^{-1/2}(r), \quad \underset{2}{\lambda}^2=h_2^{-1/2}(r), \quad \underset{3}{\lambda}^3=h_3^{-1/2}(r)\sin^{-1}\vartheta\\ \text{(all the other components of } \underset{i}{\lambda}^\mu\text{-s are vanishing)}\end{cases} \tag{4.15}$$

and on the other

$$\xi_1=h_1^{-1/2}(r)p_1/m_0,\quad \xi_2=h_2^{-1/2}(r)p_2/m_0,\quad \xi_3=h_3^{-1/2}(r)p_3/m_0\sin\vartheta, \tag{4.16}$$

respectively. This means, however, that due to eq. (4.13) the absolute value of the three-momentum may be substituted by the absolute value ξ of the inhomogeneous direction co-ordinates defined by

$$p\equiv\xi=\left\{\frac{\varepsilon^2}{m_0^2}h_0^{-1}(r)-1\right\}^{1/2} \tag{4.17}$$

and finally the momentary relativistic phase-space volume in eq. (4.7) on the $x^0=$ $=\text{const.}$ hyper-plain can be written in the from:

$$\Omega_0 = m_0^3\int_\alpha^R dr\int_0^\pi d\vartheta\int_0^{2\pi}d\varphi\int_0^{\xi_F}d\xi\int_0^\pi d\theta\int_0^{2\pi}d\Phi\{h_0h_1h_2h_3\}^{1/2}\sin\vartheta\xi^2\sin\theta=$$
$$=\frac{(4\pi)^2}{3}m_0^3\int_\alpha^R dr\{h_0h_1h_2h_3\}^{1/2}\left\{\frac{\varepsilon_F^2}{m_0^2}h_0^{-1}-1\right\}^{3/2}, \tag{4.18}$$

where ξ_F and ε_F mean the values of ξ and ε, respectively, on the Fermi level of the momentum-space. Due to the definition of the three-dimensional volume in the configuration-space, we have

$$V_0 = 4\pi \int_\alpha^R dr \{h_0 h_1 h_2 h_3\}^{1/2}, \tag{4.19}$$

and one can write instead of eq. (4.2)

$$N = \gamma \frac{(4\pi)^2}{3} \left(\frac{m_0}{2\pi\hbar} \right)^3 \int_\alpha^R dr \{h_0 h_1 h_2 h_3\}^{1/2} \left\{ \frac{\varepsilon_F^2}{m_0^2} h_0^{-1}(r) - 1 \right\}^{3/2}. \tag{4.20}$$

Owing to the mean density (4.6) of the relativistic fermion gas, its density $\varrho(r)$ on the hyper-plain $x^0 = \text{const.}$ of the configuration-space may be defined by

$$\frac{N}{V_0} = \frac{\gamma}{V_0} \frac{(4\pi)^2}{3} \left(\frac{m_0}{2\pi\hbar} \right)^3 \int_\alpha^R dr \{h_0 h_1 h_2 h_3\}^{1/2} \left\{ \frac{\varepsilon_F^2}{m_0^2} h_0^{-1}(r) - 1 \right\}^{3/2} = 4\pi \int_\alpha^R dr\, r^2 \varrho(r). \tag{4.21}$$

Indeed, let the three-dimensional density $\varrho(r)$ be introduced by

$$\varrho(r) \stackrel{\text{def}}{=} \frac{8}{V_0} \frac{4\pi}{3} \left(\frac{m_0}{2\pi\hbar} \right)^3 \frac{1}{r^2} \{h_0 h_1 h_2 h_3\}^{1/2} \left\{ \frac{\varepsilon_F^2}{m_0^2} h_0^{-1}(r) - 1 \right\}^{3/2}. \tag{4.22}$$

So far the Fermi energy of the particles in unknown and is terms of $\bar{\varrho}$ is only implicitly determined by eq. (4.21). In order to calculate it explicitly one has to carry out the r-integration in eq. (4.21)

Finally, it seems to be worthwhile to introduce the three-dimensional energy density of the fermion gas being the corresponding $T_0^0(r)$ component of the energy-momentum tensor of the system. Due to the definition of the zero point kinetic energy this may be carried out by means of the relation:

$$\begin{aligned} E_0 &= \int_\alpha^R dr \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \int_0^{\varepsilon_F} d\varepsilon\, r^2 \varrho(r) \varepsilon(r) = \\ &= 2\pi\varepsilon_F^2 \int_\alpha^R dr\, r^2 \varrho(r) = -4\pi \int_\alpha^R dr \{h_0 h_1 h_2 h_3\}^{1/2} T_0^0(r) \end{aligned} \tag{4.23}$$

based on which the definition

$$T_0^0(r) \stackrel{\text{def}}{=} -\frac{2\pi}{3} \frac{8}{V_0} \left(\frac{m_0}{2\pi\hbar} \right)^3 \varepsilon_F^2 \left\{ \frac{\varepsilon_F^2}{m_0^2} h_0^{-1}(r) - 1 \right\}^{3/2} \tag{4.24}$$

can be suggested.

We are particularly interested in the special case of relativistic neutrino gas. This means that we have to carry out the limiting process $m_0 \to 0$ in the formulae obtained above.

First of all, one observes that due to eqs. (4.6) and (4.21), furthermore, to the fact that owing to the spin degeneracy in the case of neutrinos $\gamma = 2$, the mean density of the neutrino gas on the hyper-plain $x^0 = \text{const.}$ of the configuration-space is determined in terms of

$$\bar{\varrho} = \frac{4\varepsilon_F^3}{3\pi V_0 \hbar^3} \int_\alpha^R dr\, h_0^{-1} \{h_1 h_2 h_3\}^{1/2}; \tag{4.25}$$

i. e., the Fermi energy is given by means of $\bar{\varrho}$ as

$$\varepsilon_F = \hbar \left[\frac{3\pi}{4} \bar{\varrho} \int_\alpha^R dr \{h_0 h_1 h_2 h_3\}^{1/2} \right]^{1/3} \left[\int_\alpha^R dr\, h_0^{-1} \{h_1 h_2 h_3\}^{1/2} \right]^{-\frac{1}{3}}. \tag{4.26}$$

As to the three-dimensional density (4.22) of the neutrino gas and to its three-dimensional energy density (4.23) on the hyper-plain $x^0 = \text{const.}$ of the configuration-space we have:

$$\varrho(r) = \frac{\varepsilon_F^3}{3\pi^2 V_0 \hbar^3} \frac{1}{r^2 h_0} \{h_1 h_2 h_3\}^{1/2} \tag{4.27}$$

Table I

	Einstein's universe	Schwartzschild's solution	Special solution with cylindrical symmetry
h_0	1	$1-\alpha/r$	$1-\alpha/r$
h_1	$(1-r^2/R^2)^{-1}$	$(1-\alpha/r)^{-1}$	$(1-\alpha/r)^{-1}$
$h_2 = h_3$	r^2	r^2	$r^2(1-\alpha/r)^{-1}$
$\{h_1 h_2 h_3\}^{1/2}$	$r^2(1-r^2/R^2)^{-\frac{1}{2}}$	$r^2(1-\alpha/r)^{-\frac{1}{2}}$	$r^2(1-\alpha/r)^{-\frac{3}{2}}$
α	0	$2\,M$	$2\,M$
$\varepsilon_F(R)$	—	$\hbar\left(\frac{3\pi}{4}\bar{\varrho}\right)^{1/3}\left\{1+\frac{9}{4}\frac{\alpha}{R}\right\}^{1/3}$	$\hbar\left(\frac{3\pi}{4}\bar{\varrho}\right)^{1/3}\left\{1+\frac{15}{4}\frac{\alpha}{R}\right\}^{1/3}$
ε_F	$\hbar\left(\frac{3\pi}{4}\bar{\varrho}\right)^{1/3}$	$\hbar\left(\frac{3\pi}{4}\bar{\varrho}\right)^{1/3}$	$\hbar\left(\frac{3\pi}{4}\bar{\varrho}\right)^{1/3}$
$\varrho(r)$	$\frac{\varepsilon_F^3}{3\pi^2 V_0 \hbar^3}\left\{1-\frac{r^2}{R^2}\right\}^{-\frac{1}{2}}$	$\frac{\varepsilon_F^3}{3\pi^2 V_0 \hbar^3}\left\{1-\frac{\alpha}{r}\right\}^{-\frac{3}{2}}$	$\frac{\varepsilon_F^3}{3\pi^2 V_0 \hbar^3}\left\{1-\frac{\alpha}{r}\right\}^{-\frac{5}{2}}$
$T_0^0(r)$	$\frac{\varepsilon_F^5}{6\pi^2 V_0 \hbar^3}$	$\frac{\varepsilon_F^5}{6\pi^2 V_0 \hbar^3}\left\{1-\frac{\alpha}{r}\right\}^{-\frac{3}{2}}$	$\frac{\varepsilon_F^5}{6\pi^2 V_0 \hbar^3}\left\{1-\frac{\alpha}{r}\right\}^{-\frac{3}{2}}$

and

$$T_0^0(r) = -\frac{\varepsilon_F^5}{6\pi^2 V_0 \hbar^3} h_0^{-3/2}(r), \tag{4.28}$$

respectively.

In order to obtain the final results in the case of different Riemannian space-time continuums the explicit expressions of the metrical fundamental tensor in eqs. (4.9) have to be taken into account. These are summarized in *Table I.*

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